K25P 1902

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (C.B.C.S.S. – OBE-Reg./Supple./Imp.) Examination, April 2025
(2023 and 2024 Admissions)
MATHEMATICS
MSMAT02C10/MSMAF02C10 : PDE and Integral Equations

Time : 3 Hours   Max. Marks : 80

PART – A

Answer **any five** questions. **Each** question carries **4** marks.

1. Define a quasilinear equation. Give an example.

2. When an equation $L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$ is elliptic. Give an example of an elliptic equation.

3. Let u(x, t) be the solution of the Cauchy problem

   $u_{tt} - 9u_{xx} = 0, -\infty < x < \infty, t > 0,$

   with initial conditions $$u(x,0) = f(x) = \begin{cases} 1 & \text{if } |x| \le 2, \\ 0 & \text{if } |x| > 2, \end{cases} \quad u_t(x,0) = g(x) = \begin{cases} 1 & \text{if } |x| \le 2, \\ 0 & \text{if } |x| > 2 \end{cases}.$$

   Find $u\left(0, \frac{1}{6}\right)$.

4. Define Neumann problem.

5. Define linear integral equation. Give an example.

6. Transform the differential equation with initial conditions $\left.\begin{aligned} \frac{d^2y}{dx^2} + \lambda y = f(x), \\ y(0) = 1, y'(0) = 0 \end{aligned}\right\}$ into an integral equation. **(5×4=20)**

P.T.O.

PART – B

Answer **any three** questions. **Each** question carries **7** marks.

7. Prove the following :

   $u_x = c_0 u + c_1$ where c_0 is a constant and c_1 is a function of x and y has infinitely many solutions when $c_1 = 0$ and $u(x, 0) = 2e^{c_0 x}$.

8. Find the characteristic equations of the eikonal equation

   $F(x, y, u, p, q) = p^2 + q^2 - n_0^2 = 0.$

9. Prove that the Cauchy problem $u_{tt} - c^2 u_{xx} = F(x, t)$, $-\infty < x < \infty$, $t > 0$, with initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $-\infty < x < \infty$. admits at most one solution.

10. Solve the problem $u_{tt} - 4u_{xx} = 0$, $0 < x < 1$, $t > 0$, with boundary conditions $u_x(0, t) = u_x(1, t) = 0$, $t \geq 0$, and initial conditions $u(x, 0) = f(x) = \cos^2(\pi x)$, $0 \leq x \leq 1$, $u_t(x, 0) = g(x) = \sin^2(\pi x)\cos(\pi x)$, $0 \leq x \leq 1$.

11. Prove that $\overbrace{\int_a^x \cdots \int_a^x}^{n \text{ times}} f(x) \overbrace{dx \ldots dx}^{n \text{ times}} = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi$. **(3×7=21)**

PART – C

Answer **any three** questions. **Each** question carries **13** marks.

12. a) Solve the equation $u_x + u_y = 2$ subject to the initial condition $u(x, 0) = x^2$.

    b) Consider the Cauchy problem $u_x + u_y = 1$, $u(x, x) = x$. Show that it has infinitely many solutions.

13. Prove the following result : Assume that the coefficients of the quasilinear equation $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$ are smooth functions of their variables in a neighborhood of the initial curve $x(0, s) = x_0(s)$, $y(0, s) = y_0(s)$, $u(0, s) = u_0(s)$.

Assume further that the transversality condition holds at each point s in the interval $(s_0 - 2\delta, s_0 + 2\delta)$ on the initial curve. Then prove that the Cauchy problem has a unique solution in the neighborhood $(t, s) \in (-\varepsilon, \varepsilon) \times (s_0 - \delta, s_0 + \delta)$ of the initial curve. If the transversality condition does not hold for an interval of s values, then prove that the Cauchy problem has either no solution at all, or it has infinitely many solutions.

14. a) Prove that the type of a linear second-order PDE in two variables is invariant under a change of coordinates. In other words, the type of the equation is an intrinsic property of the equation and is independent of the particular coordinate system used.

    b) Consider the Tricomi equation : $u_{xx} + xu_{yy} = 0$, $x > 0$. Find a canonical transformation $q = q(x, y)$, $r = r(x, y)$ and the corresponding canonical form.

15. a) Prove that a necessary condition for the existence of a solution to the Neumann problem is $\int_{\partial D} g(x(s), y(s))\, ds = \int_D F(x, y)\, dx\, dy$, where $(x(s), y(s))$ is a parameterization of ∂D.

    b) Let D be a planar domain, let u be a harmonic function there, and let (x_0, y_0) be a point in D. Assume that B_R is a disk of radius R centered at (x_0, y_0), fully contained in D. For any $r > 0$, set $C_r = \partial B_r$. Then prove that the value of u at (x_0, y_0) is the average of the values of u on the circle C_R :
    $$u(x_0, y_0) = \frac{1}{2\pi R}\int_{C_R} u(x(s), y(s))ds = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)d\theta.$$

16. a) Prove that the relation $y(x) = \int_a^b G(x, \xi)\, \Phi(\xi)\, d\xi$ where
    $$G(x, \xi) = \begin{cases} -\dfrac{1}{A}u(x)v(\xi) & \text{when } x < \xi, \\ -\dfrac{1}{A}u(\xi)v(x) & \text{when } x > \xi, \end{cases}$$
    implies the differential equation. $Ly + \Phi(\xi) = 0$.

    b) Prove that $y(x) = \lambda\int_0^1 (1 - 3x\xi)y(\xi)d\xi$ has only the trivial solution if and only if $\lambda \neq \pm 2$.

(3×13=39)